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The homology of the connective K-theory spectrum as an $\mathscr{A}(1)$-module

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Abstract

From an odd prime p, let l be the Adams summand of p-local connective K-theory and $\mathscr{A}(1)$ the subalgebra of the mod p Steenrod algebra generated by Q_0 and the first Steenrod power $\mathscr{P}^1$. The algebra $\mathscr{A}(1)$ is an explicitly understood Hopf algebra over $\mathbb{F}_p$ of $\mathbb{F}_p$-dimension $4p$. The mod p homology $H_*(l)$ of l is a tensor product of a polynomial algebra on countably many generators with an exterior algebra on countably many generators. We describe $H_*(l)$ as an $\mathscr{A}(1)$-module.

1. Introduction

Connective K-theory k_* is by now a well-understood generalized cohomology theory. The spectrum k of k_* localized at an odd prime p splits into $p-1$ pieces,

$$k_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} l,$$

with $l_*(S^0) = \mathbb{Z}_{(p)}[v_1]$. Here v_1 has degree $q := 2p-2$ and is the $(p-1)$th power of the usual Bott element $u \in k_2(S^0)$. The mod p homology $H_*(l)$ of l is a simple algebraic object. It is closely related to the dual $\mathscr{A}_*$ of the mod p Steenrod algebra $\mathscr{A}$: The generator $\rho \in H^0(l)$ embeds $H_*(l)$ into $\mathscr{A}_* = \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda(\tau_0, \tau_1, \ldots)$ as the following subalgebra:

$$H_*(l) = \mathbb{F}_p[\overline{\xi_1}, \overline{\xi_2}, \ldots] \otimes \Lambda(\overline{\tau_2}, \overline{\tau_3}, \ldots).$$

Here $\bar{a}$ is the image of $a \in \mathscr{A}_*$ under the canonical anti-automorphism of $\mathscr{A}_*$ and ξ_i and τ_i are the usual Milnor generators with degree$(\overline{\xi_i}) = 2p^i - 2$, and degree $(\overline{\tau_i}) = 2p^i - 1$. *Let* $E = \Lambda(Q_0, Q_1)$ be the subalgebra of the mod p Steenrod algebra $\mathscr{A}$ generated by the two Milnor elements Q_0, Q_1 and $\mathscr{A}(1)$ the subalgebra of $\mathscr{A}$

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generated by Q_0 and the first Steenrod power $\mathscr{P}^1$. We have $Q_1 = \mathscr{P}^1 Q_0 - Q_0 \mathscr{P}^1$ and in $\mathscr{A}(1)$ there are the relations

$$Q_0Q_0 = 0, \quad Q_1Q_1 = 0, \quad Q_0Q_1 = -Q_1Q_0, \quad (\mathscr{P}^1)^p = 0, \quad \mathscr{P}^1Q_1 = Q_1\mathscr{P}^1.$$

These subalgebras of $\mathscr{A}$ are quite tractable small Hopf algebras over $\mathbb{F}_p$, with $\dim_{\mathbb{F}_p} E = 4$ and $\dim_{\mathbb{F}_p} \mathscr{A}(1) = 4p$. The usual left action of $\mathscr{A}$ on $H_*(l)$ restricts to define actions of E and $\mathscr{A}(1)$ which are explicitly described by the action on generators

$$Q_0(\bar{\tau}_i) = \bar{\xi}_i, \quad Q_1(\bar{\tau}_i) = \bar{\xi}_{i-1}^p, \quad Q_0(\bar{\xi}_i) = 0, \quad Q_1(\bar{\xi}_i) = 0,$$

$$\mathscr{P}^1(\bar{\tau}_i) = 0, \quad \mathscr{P}^1(\bar{\xi}_i) = \bar{\xi}_{i-1}^p \tag{1.1}$$

noting that Q_0, Q_1 and $\mathscr{P}^1$ act as derivations.

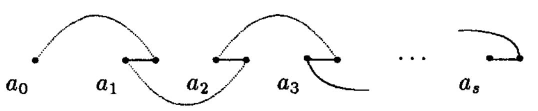
The classification of graded modules over E is well known (e.g. see [1] or [4]) and the knowledge of the E-module structure of $H_*(l)$ has led to important applications (e.g. see [1]). Despite the fact that a classification of graded $\mathscr{A}(1)$-modules seems not to be known, information on the $\mathscr{A}(1)$-module structure of $H_*(l)$ is necessary in certain applications; see the discussion at the end of this introduction. The purpose of this note is to provide this information. Our main result is a description of the $\mathscr{A}(1)$-module structure of $H_*(l)$ as a sum of tensor products of simple and explicitly known $\mathscr{A}(1)$-modules and the determination of the indecomposable $\mathscr{A}(1)$-modules which appear in $H_*(l)$.

The objects considered are completely explicit and rather independent from the setting in which they arose. Also the methods used are essentially elementary so it can be hoped that the results might be of use or interest somewhere else too. To describe the results in more detail define a weight function wt on the monomials in $H_*(l)$ by

$$\mathrm{wt}(\bar{\xi}_i) = \mathrm{wt}(\bar{\tau}_i) = p^{i-1} \quad \text{and} \quad \mathrm{wt}(a \cdot b) = \mathrm{wt}(a) + \mathrm{wt}(b) \tag{1.2}$$

and let $W(n)$ be the $\mathbb{F}_p$-vector space generated by all monomials of weight n. Then $W(n)$ is an E-module and for $n \equiv 0(p)$ an $\mathscr{A}(1)$-module. The $\mathscr{A}(1)$-module $W(sp^a)$, $1 \le s < p$, is particularly simple.

It decomposes into the pure lightning flash module $N(sp^a)$ (see Section 2) and a finite free $\mathscr{A}(1)$-module. As E-module $N(sp^a)$ may be visualized by the picture



Here a point denotes a copy of $\mathbb{F}_p$ and straight (curved) arrows indicate a non-trivial action of Q_0 (Q_1).

If $n = \sum_{i=a}^{b} n_i p^i$, $a \ge 1$, is the p-adic representation of n, we show that the multiplication in $H_*(l)$ induces an $\mathscr{A}(1)$-surjection

$$\bigotimes_{i=a}^{b} W(n_i p^i) \xrightarrow{m} W(n)$$

with finite $\mathscr{A}(1)$-free kernel. This describes $W(n)$ as a tensor product of very simple $\mathscr{A}(1)$-modules and leads to a description of the $\mathscr{A}(1)$-module structure of $H_*(l)$. In Section 6 we show that the indecomposables in $W(n)$, $n \equiv 0(p)$, consist of copies of certain submodules $E_i \subset \mathscr{A}(1)_*$, $1 \le i \le p$, where $\mathscr{A}(1)_*$ is the Hopf algebra dual to $\mathscr{A}(1)$, and a new $\mathscr{A}(1)$-module which is called lightning flash module with tag (for a picture see Section 6).

A main source of motivation to investigate the E-module structure on the mod p homology $H_*(X)$ of a space or spectrum X comes from the approach to use the classical Adams spectral sequence for computing $l_*(X)$. The E_2-term of this spectral sequence is given by the Ext-groups $\mathrm{Ext}^{s,t}_{\mathscr{A}_*}(\mathbb{F}_p, H_*(l \wedge X))$. By a standard change-of-rings isomorphism these Ext-groups simplify drastically:

$$\mathrm{Ext}^{s,t}_{\mathscr{A}_*}(\mathbb{F}_p, H_*(l \wedge X)) \cong \mathrm{Ext}^{s,t}_{E_*}(\mathbb{F}_p, H_*(X)). \tag{1.3}$$

Here $E_* = \Lambda(\tau_0, \tau_1)$ is the Hopf algebra dual to E. Since the classification of graded modules over E is known and very simple it is quite often possible to determine these Ext-groups and obtain information on $l_*(X)$ in this way. A prominent example is the computation of $l_*(l)$ and the construction of a splitting of $l \wedge l$ (e.g. see [1]).

A step closer to stable homotopy theory than l is $\mathrm{Im}(J)$-theory A_*. It may be defined by the cofibre sequence of spectra

$$\longrightarrow A \xrightarrow{D} l \xrightarrow{Q} \Sigma^q l \xrightarrow{\Delta} \Sigma A \longrightarrow \tag{1.4}$$

Here Q is the l-theory operation defined by $v_1 \cdot Q = \psi^k - 1$, where ψ^k is the stable Adams operation and k generates $(\mathbb{Z}/p^2)^*$. To investigate $A_*(X)$ in a similar way one needs $H_*(X)$ as an $\mathscr{A}(1)$-module (or equivalently as an $\mathscr{A}(1)_*$-comodule).

For the construction of a splitting of $A \wedge A$ and the computation of $A^*(A)$ in [3] the $\mathscr{A}(1)$-module structure of $H_*(l)$ and $H_*(A)$ was needed. This is my main application and the motivation for the results presented here.

Section 2 fixes notation and contains some preliminaries on $\mathscr{A}(1)$-modules. In Section 3 the action of $\mathscr{P}^1$ on $B = \mathbb{F}_p[\overline{\xi_1^p}, \overline{\xi_2}, \ldots]$ is studied. This is used in Sections 4 and 5 to derive the tensor product description of $H_*(l)$ and $H_*(A)$. Section 6 discusses the tagged lightning flash module and the decomposition into indecomposables.

Throughout the paper p denotes an odd prime, $q = 2p - 2$, k generates $(\mathbb{Z}/p^2)^*$, $v(n) = v_p(n)$ is the power of p in the prime factorization of n and $H_*(X)$ is mod p homology.

2. Preliminaries

Consider the following sub-Hopf algebras of the mod p Steenrod algebra $\mathscr{A}$. Let $\Gamma \cong \mathbb{F}_p[\mathscr{P}^1]/(\mathscr{P}^1)^p$ be the subalgebra generated by $\mathscr{P}^1$, $E = \Lambda(Q_0, Q_1)$ the subalgebra generated by Q_0 and Q_1 and $\mathscr{A}(1)$ the subalgebra generated by Q_0 and $\mathscr{P}^1$. The dual

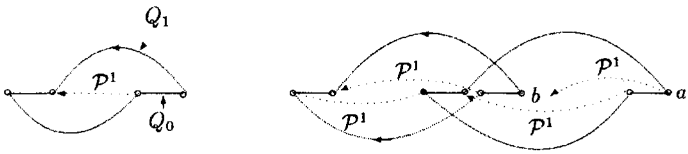
Hopf algebra of B for B in $\{\Gamma, E, \mathscr{A}(1)\}$ is denoted by B_*. We may dualize between B-modules and B-comodules in the usual way (e.g. see [1, p. 332]).

There is a unique indecomposable Γ-module V_i of $\mathbb{F}_p$-dimension i, $1 \le i \le p$. Then V_1 is irreducible and V_p is free. Now Γ-modules decompose, by Jordan decomposition, into a direct sum of V_i's. We shall need the following $\mathscr{A}(1)$-modules. Define E_i, $1 \le i \le p$, as the $\mathscr{A}(1)$-submodule of $\mathscr{A}(1)_*$ generated by $\tau_0 \cdot \tau_1 \cdot \xi_1^{i-1}$ where $\mathscr{A}(1)_* = \Lambda(\tau_0, \tau_1) \otimes \mathbb{F}_p[\xi_1]/(\xi_1^p)$ is viewed as a quotient of $\mathscr{A}_* = \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes \Lambda(\tau_0, \tau_1, \tau_2, \dots)$. Then $E_p = \mathscr{A}(1)_*$ and the $\mathscr{A}(1)$-action on E_i is given by (1.1), $Q_0(\tau_0) = -1$ and $\bar{\tau}_1 = \xi_1\tau_0 - \tau_1$, $\bar{\xi}_1 = -\xi_1$ and $\bar{\tau}_0 = -\tau_0$.

If we regard the Γ-module V_i as an $\mathscr{A}(1)$-module in the obvious way then clearly

$$E_i = E_1 \otimes^{\Delta} V_i$$

as $\mathscr{A}(1)$-modules where $\otimes^{\Delta}$ indicates the diagonal action. Then as an E-module, E_i is a direct sum of i copies of E. For example E_1 and E_2 may be visualized by the following pictures.



$(\mathcal{P}^1 a = b \quad \mathcal{P}^1 Q_0 a = Q_1 a + Q_0 b)$

Here dots represent $\mathbb{F}_p$-basis elements and the action of Q_0, Q_1 and $\mathscr{P}^1$ is indicated by lines.

Let M be a finite $\mathscr{A}(1)$-module. Since $E_i = \mathscr{A}(1)_* \,\square_{\Gamma_*}\, V_i$, one needs only to know the Γ-module structure on M in order to compute the $\mathscr{A}(1)$-module structure on $E_i \otimes^{\Delta} M$.

$$E_i \otimes^{\Delta} M \cong \mathscr{A}(1)_* \,\square_{\Gamma_*} (V_i \otimes^{\Delta} M) \tag{2.1}$$

(see [3, (1.8)]) and the Jordan decomposition of $V_i \otimes V_j$ (see Section 3).

The pure lightning flash module N^m is the $\mathbb{F}_p$-vector space with basis $a_0, a_i, \tilde{a}_i$, $i = 1, \dots, m$ in degrees $|a_i| = iq$, $|\tilde{a}_i| = iq + 1$ and $\mathscr{A}(1)$-module structure $Q_0(\tilde{a}_i) = a_i$, $Q_1(\tilde{a}_i) = a_{i-1}$, $\mathscr{P}^1(a_i) = i \cdot a_{i-1}$, $\mathscr{P}^1(\tilde{a}_i) = (i-1)\tilde{a}_{i-1}$.

It is convenient to introduce the notation $N(n)$ for $N^{\nu(n!)}$. A picture of $N(n)$ was given in Section 1.

Invariants for $\mathscr{A}(1)$-modules are given by Q_0, Q_1 and $\mathscr{P}^i$-homology groups: For a graded $\mathscr{A}(1)$-module M, $H_*(M; Q_0), H_*(M; Q_1)$ are defined as usual and $\mathscr{P}^i$-homology groups are defined by

$$H_n(M; \mathscr{P}^i) = \frac{\ker((\mathscr{P}^1)^i : M_n \to M_{n-qi})}{\operatorname{im}((\mathscr{P}^1)^{p-i} : M_{n+(p-i)q} \to M_n)}.$$

Some elementary properties of $\mathscr{P}^i$-homology groups are given in [5]. We shall need the following facts. A short exact sequence $0 \to M' \to M \to M'' \to 0$ of Γ-modules induces a long exact sequence of $\mathscr{P}^i$-homology groups

$$\longrightarrow H_*(M';\mathscr{P}^i) \longrightarrow H_*(M;\mathscr{P}^i) \longrightarrow H_*(M'';\mathscr{P}^i) \longrightarrow H_*(M';\mathscr{P}^{p-i}) \longrightarrow \tag{2.2}$$

and a Γ-module map $f: M \to N$ induces an isomorphism $f_*: H_n(M;\mathscr{P}^i) \to H_n(N;\mathscr{P}^i)$ for all n and $i = 1$ if and only if f_* is an isomorphism for all n and all $i \in \{1, \dots, p-1\}$.

Standard notations and results well known for E-modules (e.g. see [1, 5, 4]) are transferred to $\mathscr{A}(1)$-modules without much change. For example, there is a category of stable $\mathscr{A}(1)$-modules: Two $\mathscr{A}(1)$-modules M, N are stably isomorphic if there exist free modules F_1, F_2 such that $M \oplus F_1 \cong M \oplus F_2$. We have the well-known

Proposition 2.3. Let M be a graded $\mathscr{A}(1)$-module which is bounded below and has vanishing Q_0, Q_1 and $\mathscr{P}^1$-homology groups. Then M is free.

Proof. Since $\mathscr{A}(1)$ is injective [4], it is enough to find a copy of $\mathscr{A}(1)$ in M. Let x be an element of lowest degree. The assumptions then easily imply that there is an element y with $(\mathscr{P}^1)^{p-1} \circ Q_0 \circ Q_1(y) = x$. Hence y generates a free $\mathscr{A}(1)$-submodule of M. $\square$

Corollary 2.4. Let $f: M \to N$ be a map of finite-dimensional $\mathscr{A}(1)$-modules. If f induces an isomorphism in Q_0, Q_1 and $\mathscr{P}^1$-homology groups, then f is a stable equivalence.

Proof. The standard proof [1] applies, using the long exact sequences in Q_0, Q_1 and $\mathscr{P}^i$-homology. $\square$

Recall also that in the category of finite $\mathscr{A}(1)$-modules there exists a decomposition into indecomposable modules which is unique up to isomorphism. A good reference is [4, Chs. 11, 12]. In particular, there are no non-cancellation phenomena, i.e.

$$\text{if } F \text{ is free and } A \oplus F = B \oplus F \quad \text{then } A \cong B. \tag{2.5}$$

(Of course, there is an easy direct proof for this.) Hence any stable decomposition into indecomposables implies an unstable one.

3. The Γ-module structure of B

As a preparation for the investigation of the $\mathscr{A}(1)$-module structure of $H_*(l)$ we study in this section the Γ-module structure of $B = \mathbb{F}_p[\overline{\xi_1^p}, \overline{\xi_2}, \dots]$. Recall Γ is $\mathbb{F}_p[\mathscr{P}^1]/(\mathscr{P}^1)^p$ and that V_i denotes the indecomposable Γ-module of $\mathbb{F}_p$-dimension i, $i \in \{1, \dots, p\}$. Since Γ is isomorphic to the group ring $\mathbb{F}_p[\mathbb{Z}/p]$, Γ-modules may also be viewed as modular representations of $\mathbb{Z}/p$. In this setting there is a standard action of

$\mathbb{F}_p[\mathbb{Z}/p]$ on a tensor product $V \otimes W$ ($\mathbb{Z}/p$ with generator g acts as $g(v \otimes w) = g(v) \otimes g(w)$). With respect to this action, the Jordan decomposition of $V_i \otimes V_j$ is well known (see e.g. [6, 2]).

Theorem 3.1. *Assume $1 \le r \le s \le p$ and let*

$$c = \begin{cases} r & \text{if } r + s \le p, \\ p - s & \text{if } r + s \ge p. \end{cases}$$

Then as $\mathbb{F}_p[\mathbb{Z}/p]$-modules

$$V_r \otimes V_s \cong \bigoplus_{j=0}^{r-c-1} V_p \oplus \bigoplus_{k=1}^{c} V_{s-r+2k-1}.$$

This result is a direct consequence of the following formula:

$$V_2 \otimes V_n \cong V_{n+1} \oplus V_{n-1} \quad \text{for } n < p. \tag{3.2}$$

Now $\mathscr{P}^1$ is acting as a derivation, so the induced action on $V \otimes W$ is different from the one used here. Nevertheless formula (3.2) is also true for Γ-modules with $\mathscr{P}^1$ acting as a derivation: If V_2 has $\mathbb{F}_p$-basis $a, \mathscr{P}^1(a)$ and V_n has $\mathbb{F}_p$-basis $b, \mathscr{P}^1(b), (\mathscr{P}^1)^2(b), \ldots$ then $a \otimes b$ generates a copy of V_{n+1} and

$$a \square_n b := a \otimes \mathscr{P}^1 b + (1 - n)\mathscr{P}^1 a \otimes b \tag{3.3}$$

generates a copy of V_{n-1} as Γ-modules.

Therefore the decomposition of $V_r \otimes V_s$ as Γ-module is also given by Theorem 3.1. If all modules are graded and $\mathscr{P}^1$ acts as a map of degree $-q$, then

$$V_2 \otimes V_n \cong V_{n+1} \oplus \Sigma^{-q} V_{n-1}$$

and

$$V_r \otimes V_s \cong \bigoplus_{j=0}^{r-c-1} \Sigma^{-qj} V_p \oplus \bigoplus_{k=1}^{c} \Sigma^{-q(k-r)} V_{s-r+2k-1}.$$

To simplify the notation we shall suppress the grading in writing down the decomposition of a Γ-module.

Every Γ-module has a canonical filtration $F^i M := \ker(\mathscr{P}^1)^i$,

$$0 = F^0 M \subset F^1 M \subset F^2 M \subset \cdots \subset F^p M = M.$$

Define $\bar{M}^i := (F^i M / F^{i-1} M) / \mathscr{P}^1 (F^{i+1} M / F^i M)$. If $\{b_j\}$ is an $\mathbb{F}_p$-basis of $\bar{M}^i$, then every b_j generates a cyclic submodule of M isomorphic to V_i, hence $\{b_j\}$ generates a submodule M^i such that $M \cong M^1 \oplus M^2 \oplus \cdots \oplus M^p$. Observe that for $i < p$: $\bar{M}^i \subset H_*(M; \mathscr{P}^{p-1})$ and $(\mathscr{P}^1)^{i-1} \bar{M}^i \subset H_*(M; \mathscr{P}^1)$ so that $H_*(M; \mathscr{P}^{p-1}) \cong \bar{M}^1 \oplus \bar{M}^2 \oplus \cdots \oplus \bar{M}^{p-1}$ and $H_*(M; \mathscr{P}^1) \cong \bar{M}^1 \oplus \mathscr{P}^1(\bar{M}^2) \oplus \cdots \oplus (\mathscr{P}^1)^{p-2}(\bar{M}^{p-1})$.

Let $B = \mathbb{F}_p[\zeta_1^p, \zeta_2, \zeta_3, \ldots]$, $\zeta_i := \bar{\xi}_i$ with Γ-action given by $\mathscr{P}^1 \zeta_i = \zeta_{i-1}^p$. To describe $H_*(B; \mathscr{P}^i)$ we introduce the following notation: For two indeterminates $A, \mathscr{P}^1 A$ let

$C(A)$ be the $\mathbb{F}_p$-vector space with basis $1, A, \mathscr{P}^1 A, \dots, A^i \cdot (\mathscr{P}^1 A)^j$, $i + j \le p - 2$ and Γ-action generated by $A \mapsto \mathscr{P}^1 A$ with $\mathscr{P}^1 \circ \mathscr{P}^1 A = 0$ and $\mathscr{P}^1$ acting as a derivation. Then $C(A)$ has $\mathbb{F}_p$-dimension $p(p-1)/2$ and decomposes as a Γ-module as $V_1 \oplus V_2 \oplus \cdots \oplus V_{p-1}$.

Theorem 3.4. $H_*(B; \mathscr{P}^i) = H_*(C(\zeta_2) \otimes C(\zeta_3) \otimes \cdots; \mathscr{P}^i)$.

This result is also known to J.H. Palmieri and H. Miller.

The proof given here is an elementary induction on the number of variables. Denote $\mathbb{F}_p[\zeta_1^p, \zeta_2, \dots, \zeta_r]$ by $G(r)$ for $r \ge 2$. Then Theorem 3.4 will follow from

Proposition 3.5.

$$h_r : H_*(C(\zeta_2) \otimes C(\zeta_3) \otimes \cdots \otimes C(\zeta_r) \otimes \mathbb{F}_p[\zeta_r^p]; \mathscr{P}^i) \longrightarrow H_*(G(r); \mathscr{P}^i)$$

is an isomorphism where $h_r(\zeta_i) = \zeta_i$ *and* $h_r(\mathscr{P}^1 \zeta_i) = \zeta_{i-1}^p$.

Proof. This will be proved by induction on r. Let $\mathscr{P}^1$ act on $\mathbb{F}_p[A, \mathscr{P}^1 A]$ as a derivation satisfying $\mathscr{P}^1 \circ \mathscr{P}^1 A = 0$. The start of the induction is handled in

Lemma 3.6. $H_*(\mathbb{F}_p[A, \mathscr{P}^1 A]; \mathscr{P}^i) \cong H_*(C(A) \otimes \mathbb{F}_p[A^p]; \mathscr{P}^i)$ *induced by the inclusion map.*

Proof. Let $A^i \cdot (\mathscr{P}^1 A)^j$ have degree $i + j$ and let $S^n(A, \mathscr{P}^1 A)$ be the $\mathbb{F}_p$-vector space of homogeneous polynomials of degree n in A and $\mathscr{P}^1 A$. Then $\mathbb{F}_p[A, \mathscr{P}^1 A] = \bigoplus_n S^n(A, \mathscr{P}^1 A)$ as Γ-module. Write $n = \bar{n}p + \underline{n}$ with $0 \le \underline{n} < p$, then

$$S^n(A, \mathscr{P}^1 A) = S^{\underline{n}}(A, \mathscr{P}^1 A) \cdot (A^p)^{\bar{n}} \oplus \text{free } \Gamma\text{-modules on } A^{lp-1} \cdot (\mathscr{P}^1 A)^{n-lp+1}$$

as is easily seen. Since $C(A) = S^0(A, \mathscr{P}^1 A) \oplus S^1(A, \mathscr{P}^1 A) \oplus \cdots \oplus S^{p-2}(A, \mathscr{P}^1 A)$ and $V_p = S^{p-1}(A, \mathscr{P}^1 A)$ is free, the conclusion follows. $\square$

Lemma 3.7. *Let M be a Γ-module and let $\mathscr{P}^1$ act trivially on $\mathbb{F}_p[c]$, then*

$$H_*(M \otimes \mathbb{F}_p[c]; \mathscr{P}^i) \cong H_*(M; \mathscr{P}^i) \otimes \mathbb{F}_p[c].$$

Proof. Obvious. $\square$

For the induction step from r to $r + 1$ consider the map

$$g : G(r) \otimes \mathbb{F}_p[c, \mathscr{P}^1 c] \longrightarrow G(r+1)$$

with $g(c) = \zeta_{r+1}$, $g(\mathscr{P}^1 c) = \zeta_r^p$ and $g|_{G(r)} = \mathrm{id}$. Then

Lemma 3.8. $\ker g = (\mathscr{P}^1 c - \zeta_r^p) \cdot G(r) \otimes \mathbb{F}_p[c, \mathscr{P}^1 c]$.

Proof. By direct inspection. □

So we have a short exact sequence of Γ-modules

$$0 \longrightarrow G(r) \otimes \mathbb{F}_p[c, \mathscr{P}^1 c] \xrightarrow{i} G(r) \otimes \mathbb{F}_p[c, \mathscr{P}^1 c] \xrightarrow{g} G(r+1) \longrightarrow 0$$

inducing long exact sequences in $\mathscr{P}^i$-homology, where i is the map given by multiplication with $(\mathscr{P}^1 c - \zeta_r^p)$. Then

Lemma 3.9. *i_* is injective on $\mathscr{P}^1$- and $\mathscr{P}^{p-1}$-homology.*

Proof. It is easily seen that multiplication by $[\mathscr{P}^1 c]^p$ induces the zero map in $\mathscr{P}^i$-homology, hence multiplication by $[\mathscr{P}^1 c - \zeta_r^p]^p = [\mathscr{P}^1 c]^p - [\zeta_r^{p^2}]$ is injective (Lemma 3.7) and this implies that i_* is injective. □

Therefore we obtain short exact sequences ($i = 1, p-1$):

$$0 \longrightarrow H_*(G(r) \otimes \mathbb{F}_p[c, \mathscr{P}^1 c]; \mathscr{P}^i) \xrightarrow{i_*} H_*(G(r) \otimes \mathbb{F}_p[c, \mathscr{P}^1 c]; \mathscr{P}^i)$$

$$\xrightarrow{g_*} H_*(G(r+1); \mathscr{P}^i) \longrightarrow 0.$$

Let $M = C(\zeta_2) \otimes C(\zeta_3) \otimes \cdots \otimes C(\zeta_r) \otimes C(c)$, then by induction hypothesis we have

$$H_*(G(r) \otimes \mathbb{F}_p[c, \mathscr{P}^1 c]; \mathscr{P}^i) \cong H_*(M; \mathscr{P}^i) \otimes \mathbb{F}_p[c^p, \zeta_r^p].$$

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_*(G(r) \otimes \mathbb{F}_p[c, \mathscr{P}^1 c]; \mathscr{P}^i) \arrow[rr, "i_*"] & & H_*(G(r) \otimes \mathbb{F}_p[c, \mathscr{P}^1 c]; \mathscr{P}^i) \arrow[r, "g_*"] & H_*(G(r+1); \mathscr{P}^i) \arrow[r] & 0 \\
& H_*(M; \mathscr{P}^i) \otimes \mathbb{F}_p[c^p, \zeta_r^p] \arrow[u, "\cong"', "h_r \otimes 1"] \arrow[rr, "\cdot[\mathscr{P}^1 c - \zeta_r^p]"] & & H_*(M; \mathscr{P}^i) \otimes \mathbb{F}_p[c^p, \zeta_r^p] \arrow[u, "\cong"', "h_r \otimes 1"] \arrow[ur, "F := g_* \circ h_r \otimes 1"'] & &
\end{tikzcd}$$

Given $[x] \in H_*(M; \mathscr{P}^i) \otimes \mathbb{F}_p[c^p, \zeta_r^p]$ write $x = \sum x_j \zeta_r^{pj}$ with $[x_j]$ in $H_*(M; \mathscr{P}^i) \otimes \mathbb{F}_p[c^p]$. Now $x_j \cdot \zeta_r^{pj}$ may be written as

$$x_j \cdot \zeta_r^{pj} = x_j \cdot [\mathscr{P}^1 c]^j + x_j' \cdot [\mathscr{P}^1 c - \zeta_r^p]$$

for some x_j'. But $[x_j] \cdot [\mathscr{P}^1 c]^j$ is in $H_*(M; \mathscr{P}^i) \otimes \mathbb{F}_p[c^p]$, hence $F = g_* \circ h_r \otimes 1$ restricted to $H_*(M; \mathscr{P}^i) \otimes \mathbb{F}_p[c^p] \subset H_*(M; \mathscr{P}^i) \otimes \mathbb{F}_p[c^p, \zeta_r^p]$ is still onto. It is also easy to see that this restriction of F is injective too. This identifies the cokernel of i_* with

$$H_*(C(\zeta_2)\otimes C(\zeta_3)\otimes \cdots \otimes C(\zeta_{r+1});\mathscr{P}^i)\otimes \mathbb{F}_p[\zeta_{r+1}^p].$$

This finishes the induction step and Proposition 3.5 is proved. $\square$

Remark. Since $C(\zeta_i) = V_1 \oplus V_2 \oplus \cdots \oplus V_{p-1}$ (up to grading), Theorem 3.1 allows to work out the decomposition of $C(\zeta_2)\otimes C(\zeta_3)\otimes \cdots$ as a Γ-module. Denote by R the sum $V_1 \oplus V_2 \oplus \cdots \oplus V_{p-1}$. Then, for example,

if $p = 3$, $R\otimes R = 2R + V_p$,

if $p = 5$, $R\otimes R = 4R + 2S \bmod V_p$ ($S = V_2 + V_3$, $S\otimes R = 2R + 2S \bmod V_p$).

In particular, for $p = 3$, $C(\zeta_2)\otimes C(\zeta_3)\otimes \cdots \otimes C(\zeta_r)$ has exactly 2^r non-zero $\mathscr{P}^{p-1}$-homology classes which are represented by

$$\zeta_{i_1}\otimes\zeta_{i_2}\,\square\,\zeta_{i_3}\otimes\zeta_{i_4}\,\square\,\cdots\otimes\zeta_{i_s}\quad (s\le r),$$

where $\otimes$ and $\square = \square_2$ are the products from (3.2). For larger primes the explicit description of the homology classes in $H_*(B;\mathscr{P}^i)$ is more involved.

The weight filtration wt on $H_*(l)$ (Section 1) restricts to a weight filtration on $B = \mathbb{F}_p[\zeta_1^p, \zeta_2, \ldots]$. By definition

$$\mathrm{wt}(\zeta_a) = p^{a-1} \quad\text{and}\quad \mathrm{wt}(xy) = \mathrm{wt}(x) + \mathrm{wt}(y).$$

Let $W^B(n)$ denote the $\mathbb{F}_p$-vector space with basis consisting of all monomials of weight n. Since $\mathscr{P}^1$ respects weight, $W^B(n)$ is a Γ-submodule of B. Theorem 3.4 gives some restrictions for the $\mathscr{P}^i$-homology of $W^B(n)$.

The image of $C(\zeta_a)$ in B consists of linear combinations of monomials $\zeta_a^i\zeta_{a-1}^{pj}$ with $0 \le i + j \le p-2$, hence any non-trivial homology class in $H_*(B;\mathscr{P}^i)$ is by Theorem 3.4 a sum of monomials of the form

$$x = \zeta_1^{pt_2}\zeta_2^{s_2+pt_3}\zeta_3^{s_3+pt_4}\cdots\zeta_{r-1}^{s_{r-1}+pt_r}\zeta_r^{s_r}\quad\text{with } 0 \le s_i + t_i \le p-2. \tag{3.10}$$

The weight filtration of such a monomial is

$$(s_2 + t_2)p + (s_3 + t_3)p^2 + \cdots + (s_r + t_r)p^{r-1}.$$

Hence

Corollary 3.11. *Every non-zero $\mathscr{P}^{p-1}$-homology class y in $H^*(B;\mathscr{P}^{p-1})$ is a sum of monomials y_s of weight $\mathrm{wt}(y_s) = \sum\alpha_i p^i$ with $0 \le \alpha_i \le p-2$.*

Corollary 3.12. *$W^B((p-1)p^a)$ has trivial $\mathscr{P}^{p-1}$-homology and $W^B(sp^a)$, $0 < s \le p-2$, has exactly one non-trivial $\mathscr{P}^{p-1}$-homology class represented by ζ_{a+1}^s.*

Proof. The first part follows from Corollary 3.11 and for the second part we have $(\mathscr{P}^1)^{p-1}\zeta_{a+1}^s = 0$ since $s < p-1$ and $\zeta_{a+1}^s \notin \mathrm{im}(\mathscr{P}^1)$ since ζ_{a+1}^s is of maximal dimension in $W^B(sp^a)$. Hence ζ_{a+1}^s represents a non-trivial element in $H_*(B;\mathscr{P}^{p-1})$. Let y be another $\mathscr{P}^{p-1}$-homology class in $W^B(sp^a)$. Then y is a sum of terms x_s as in (3.10) with each x_j of weight $\sum(s_i+t_i)p^{i-1}$, $s_i+t_i \le p-2$. But $\sum(s_i+t_i)p^{i-1} = sp^a$ implies $s_i+t_i=0$ for $i \ne a+1$ and $s_{a+1}+t_{a+1}=s$. Hence x_j belongs to the $\mathscr{P}^1$-orbit of ζ^s_{a+1}. □

Finally we study the consequences of these results for the weight filtration on $H_*(l)$ itself.

Proposition 3.13. *For $a \ge 1$ the Γ-module $W(sp^a)$ has exactly 1 (for $s=p-1$) or 2 (for $s<p-1$) non-trivial $\mathscr{P}^{p-1}$-homology classes. These are represented by $\bar{\xi}^s_{a+1}$ and $\bar{\xi}^{s-1}_{a+1}\cdot\bar{\tau}_{a+1}$.*

Proof. Suppose $x=\sum\alpha_{I,J}\bar{\xi}_I\otimes\bar{\tau}_J$ represents a non-trivial $\mathscr{P}^{p-1}$-homology class of $W(sp^a)$. Then $x_K=\sum_{J=K}\alpha_{I,J}\bar{\xi}_I$ represents a non-trivial element in $H_*(B;\mathscr{P}^{p-1})$. By Corollary 3.11 we have $\mathrm{wt}(\bar{\xi}_I)=\sum\alpha_i p^{i-1}$ with $\alpha_i\in\{0,\dots,p-2\}$. Write $\mathrm{wt}(\bar{\tau}_J)=\sum\varepsilon_i p^{i-1}$ with $\varepsilon_i\in\{0,1\}$. Then $\mathrm{wt}(\bar{\xi}_I\otimes\bar{\tau}_J)=\sum(\alpha_i+\varepsilon_i)p^{i-1}$ and $\alpha_i+\varepsilon_i\le p-1$. This implies $\alpha_i=\varepsilon_i=0$ for $i\ne a+1$ and $\alpha_{a+1}+\varepsilon_{a+1}=s$ and the result follows from Corollary 3.12. □

Corollary 3.14. *The inclusion of Γ-modules*

$$\Lambda(\bar{\tau}_{a+1})\otimes C(\bar{\xi}_{a+1})\to 1\oplus W(p^a)\oplus W(2p^a)\oplus\cdots\oplus W((p-1)p^a)$$

induces an isomorphism in $\mathscr{P}^i$-homology.

Corollary 3.15. *For $n=\sum_{i=a}^b n_i p^i$, $a\ge 1$, $n_i\le p-1$, the multiplication map*

$$W(n_a p^a)\otimes W(n_{a+1}p^{a+1})\otimes\cdots\otimes W(n_b p^b)\xrightarrow{m} W(n)$$

induces an isomorphism in $\mathscr{P}^i$-homology.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\bigotimes_{a\ge 1}(1\oplus W(p^a)\oplus W(2p^a)\oplus\cdots\oplus W((p-1)p^a)) \arrow[r, "m"] & C:=\bigoplus_{n\equiv 0(p)} W(n)\subset H_*(l) \\
\bigotimes_{a\ge 1}\Lambda(\bar{\tau}_{a+1})\otimes C(\bar{\xi}_{a+1}) \arrow[u, "i"] \arrow[ur, "j"'] &
\end{tikzcd}$$

Since $C=B\otimes\Lambda(\bar{\tau}_2,\bar{\tau}_3,\dots)$ and $\mathscr{P}^1$ acts trivially on $\bar{\tau}_j$, the map j induces an isomorphism in $\mathscr{P}^i$-homology by Theorem 3.4. Corollary 3.14 implies that i induces an isomorphism in $\mathscr{P}^i$-homology, hence m induces an isomorphism in $\mathscr{P}^i$-homology as well. Extend the weight filtration to $\bigotimes_{a\ge1}(1\oplus W(p^a)\oplus W(2p^a)\oplus\cdots\oplus W((p-1)p^a))$ in the obvious way, then the subspace of weight n is $W(n_a p^a)\otimes\cdots\otimes W(n_b p^b)$ and separating by weight gives the result. □

4. $H_*(l)$ as $\mathscr{A}(1)$-module

The decomposition $H_*(l) \cong \bigoplus_n W(n)$ is a decomposition of E-modules, only for $n \equiv 0(p)$ is $W(n)$ an $\mathscr{A}(1)$-module. Since $\bar{\xi}_1$ is the only monomial of weight 1 we see $W(mp+i) \cong W(mp) \cdot \bar{\xi}_1^i$ for $0 \le i < p$ and this implies

$$H_*(l) \cong \bigoplus_m W(mp) \otimes^A V_p$$

is an isomorphism of $\mathscr{A}(1)$-modules. Thus to study $H_*(l)$ as $\mathscr{A}(1)$-module it is enough to consider $W(mp)$. We start with $W(n)$ for $n = sp^a$.

Proposition 4.1. *For $0 < s < p$, and $a \ge 1$, the $\mathscr{A}(1)$-module $W(sp^a)$ is the direct sum of the pure lightning flash module N^m, $m = \nu(sp^a!)$, and a finite free $\mathscr{A}(1)$-module $F(sp^a)$.*

Proof. Let $n := sp^a$ and $m := \nu(n!)$. We first define an $\mathscr{A}(1)$-module map

$$s_n : \Sigma^{qn} N^m \to W(n).$$

Select elements $b_i, \tilde{b}_i$ in $W(n)$ as follows:

(i) Start with $\tilde{b}_m := \bar{\tau}_{a+1}\bar{\xi}_{a+1}^{s-1}$, $b_m := \bar{\xi}_{a+1}^s$, $b_{m-1} := Q_1(\tilde{b}_m) = s^{-1} \cdot \mathscr{P}^1(b_m) = m^{-1} \cdot \mathscr{P}^1(b_m)$. Then b_m represents the only non-trivial Q_1-homology class of $W(n)$. Inductively assume $b_i, \tilde{b}_i$ for $j \ge i$ and b_{i-1} have already been defined, satisfying $\mathscr{P}^1(\tilde{b}_j) = (j-1)\tilde{b}_{j-1}$, $\mathscr{P}^1(b_j) = jb_{j-1}$, $Q_0(\tilde{b}_j) = b_j$ and $Q_1(\tilde{b}_j) = b_{j-1}$.

(ii) If $i - 1 \not\equiv 0(p)$ define $\tilde{b}_{i-1} := \mathscr{P}^1(\tilde{b}_i)/(i-1)$, $b_{i-2} := Q_1(\tilde{b}_{i-1})$. Then $(i-1) \cdot Q_0(\tilde{b}_{i-1}) = Q_0 \circ \mathscr{P}^1(\tilde{b}_i) - \mathscr{P}^1 \circ Q_0(\tilde{b}_i) + \mathscr{P}^1 \circ Q_0(\tilde{b}_i) = -Q_1(\tilde{b}_i) + \mathscr{P}^1(b_i) = -b_{i-1} + i \cdot b_{i-1} = (i-1) \cdot b_{i-1}$ and $\mathscr{P}^1(b_{i-1}) = \mathscr{P}^1 \circ Q_1(\tilde{b}_i) = Q_1 \circ \mathscr{P}^1(\tilde{b}_i) = Q_1((i-1) \cdot \tilde{b}_i) = (i-1) \cdot b_{i-2}$.

(iii) If $i - 1 \equiv 0(p)$ there is some choice. Recall that $W(n)$ has only one non-trivial Q_0-homology class which is represented by $\bar{\xi}_1^n$. We have $Q_0(b_{i-1}) = Q_0 Q_1(\tilde{b}_i) = -Q_1 Q_0(\tilde{b}_i) = -Q_1(b_i) = -Q_1 Q_1(\tilde{b}_{i+1}) = 0$. Hence as long as b_{i-1} does not represent the non-trivial Q_0-homology class, there is a preimage $\tilde{b}_{i-1}$ for b_{i-1} under Q_0. In our case b_{i-1} is a monomial in $\bar{\xi}_i$'s and we shall fix a choice for $\tilde{b}_{i-1}$ by substituting $\bar{\tau}_k$ for $\bar{\xi}_k$ where $\bar{\xi}_k$ is a factor of b_{i-1} with maximal index k. Define $b_{i-2} := Q_1(\tilde{b}_{i-1})$. We have to show $\mathscr{P}^1(b_{i-1}) = 0$ and $\mathscr{P}^1(\tilde{b}_i) = 0$.

(iv) If $m - i \ge p$ then $\tilde{b}_i$ is in the image of $(\mathscr{P}^1)^{p-1}$ and $\mathscr{P}^1(\tilde{b}_i) = 0$ follows from $(\mathscr{P}^1)^p = 0$. The same applies for b_{i-1}.

(v) If $m - i < p$, then $i = s$ and up to non-zero constants b_{i-1} is $(\mathscr{P}^1)^s(\bar{\xi}_{a+1}^s)$ and $\tilde{b}_i$ is $(\mathscr{P}^1)^{s-1}(\bar{\tau}_{a+1} \cdot \bar{\xi}_{a+1}^{s-1})$ which are in $\ker(\mathscr{P}^1)$.

The induction stops if we arrive at $b_0 = \bar{\xi}_1^n$.

Define now $s_n(a_i) := b_i$, $s_n(\tilde{a}_i) := \tilde{b}_i$. Then clearly

$$s_n : \Sigma^{qn} N^m \to W(n)$$

is an $\mathscr{A}(1)$-monomorphism.

Remark 4.2. It is only the last step (v) above which goes wrong for general n. If the p-adic representation of n is $\sum_{i=a}^{b} n_i p^i$ with $\sum n_i \geq p$ then $b_m = \bar{\xi}_{a+1}^{n_a} \cdot \bar{\xi}_{a+2}^{n_{a+1}} \cdot \cdots \cdot \bar{\xi}_{b+1}^{n_b} + Q_1(z)$ (since this represents the non-trivial Q_1-homology class of $W(n)$, $m = \sum m_i p^i := v(n!)$) and $\mathscr{P}^1(b_{m-m_0})$ is no longer zero. So every E-lightning flash in $W(n)$ is not closed under the action of $\mathscr{P}^1$. However, we shall see that there is a choice which is not closed only at the upper end.

Next we show: The cokernel of $s_n : \Sigma^{qn} N^m \to W(n)$ is a finite free $\mathscr{A}(1)$-module. Clearly s_n induces an isomorphism in Q_0- and Q_1-homology. N^m has one (if $s = p-1$) or two $\mathscr{P}^{p-1}$-homology classes which are represented by a_m and $\tilde{a}_m$, $m = v(n!)$. By Proposition 3.13 s_n induces an isomorphism in $\mathscr{P}^i$-homology as well. Hence the long exact Q_0, Q_1 or $\mathscr{P}^i$-homology sequences induced by

$$0 \to \Sigma^{qn} N^m \to W(n) \to \operatorname{cok}(s_n) \to 0$$

show that $\operatorname{cok}(s_n)$ has vanishing $Q_0, Q_1, \mathscr{P}^i$-homology, so is $\mathscr{A}(1)$-free by Proposition 2.3, proving the claim.

Write $\Sigma^{qn} F(n)$ for this cokernel and choose an $\mathscr{A}(1)$-section $y_n : \Sigma^{qn} F(n) \to W(n)$ for the projection of $W(n)$ onto $\operatorname{cok}(s_n) = \Sigma^{qn} F(n)$. Recall $N(m) = N^{v(n!)}$ and let $N_{\otimes}(n) := N(n) \oplus F(n)$ and define

$$s_n^{\otimes} := s_n + y_n : \Sigma^{qn}(N(n) \oplus F(n)) = \Sigma^{qn} N_{\otimes}(n) \to W(n). \tag{4.3}$$

Then $s_n^{\otimes}$ is an isomorphism of $\mathscr{A}(1)$-modules, finishing the proof of Proposition 4.1. □

Let now $n = \sum_{i=a}^{b} n_i p^i$, $a \geq 1$, $0 \leq n_i < p$, and define $s_n^{\otimes}$ to be the following composition:

$$s_n^{\otimes} : \Sigma^{qn} N_{\otimes}(n) := \Sigma^{qn}\left(\bigotimes_{i=a}^{b} N_{\otimes}(n_i p^i) \right) \xrightarrow{s} \bigotimes_{i=a}^{b} W(n_i p^i) \xrightarrow{m} W(n), \tag{4.4}$$

where $s = s^{\otimes}_{n_a p^a} \otimes s^{\otimes}_{n_{a+1} p^{a+1}} \otimes \cdots \otimes s^{\otimes}_{n_b p^b}$ and m is given by multiplication in the ring $H_*(l)$. Observe that m is trivially surjective, but in general not injective.

Proposition 4.5. *The kernel of* $m : W(n_a p^a) \otimes \cdots \otimes W(n_b p^b) \to W(n)$ *is a finite free* $\mathscr{A}(1)$*-module.*

Corollary 4.6. *For* $n \equiv 0(p)$, $s_n^{\otimes} : \Sigma^{qn} N_{\otimes}(n) \to W(n)$ *is a surjective* $\mathscr{A}(1)$*-map with finite free kernel.*

Proof of Proposition 4.5. The Künneth formula for Q_0- and Q_1-homology shows that m induces an isomorphism in Q_0- and Q_1-homology. From Corollary 3.15 we know that m induces an isomorphism in $\mathscr{P}^i$-homology. The long exact $Q_0, Q_1, \mathscr{P}^i$-homology sequences show that $\ker(m)$ has vanishing $Q_0, Q_1, \mathscr{P}^i$-homology, hence is free by Proposition 2.3. □

Remark. Corollary 4.6 is a description of the stable $\mathscr{A}(1)$-module structure of $W(n)$ as a tensor product of simpler and explicitly known $\mathscr{A}(1)$-modules. Since we can split off the kernel of $s_n^{\otimes}$ we also have a description of the $\mathscr{A}(1)$-module structure of $W(n)$. More information on the internal $\mathscr{A}(1)$-module structure may be found in Section 6.

5. $H_*(A)$ as $\mathscr{A}(1)$-module

The description of the mod p homology of the Im(J)-theory spectrum A (e.g. see [3, Section 2]) is obtained from the long exact sequence

$$\longrightarrow H_*(\Sigma^{q-1}l) \xrightarrow{\Delta_*} H_*(A) \xrightarrow{D_*} H_*(l) \longrightarrow \tag{5.1}$$

Define $C := \mathbb{F}_p[\bar{\xi}_1^p, \bar{\xi}_2, \ldots] \otimes \Lambda(\bar{\tau}_2, \bar{\tau}_3, \ldots) = \mathscr{A}_* \square_{\mathscr{A}(1)_*} \mathbb{F}_p \subset H_*(l)$, then (5.1) reduces to

$$0 \longrightarrow C \cdot \Delta_*(\bar{\xi}_1^{p-1}) \longrightarrow H_*(A) \xrightarrow{D_*} C \longrightarrow 0. \tag{5.2}$$

The results of [7] (see also [3]) imply that there is a ring isomorphism

$$H_*(A) \cong \tilde{C} \oplus \tilde{C} \cdot \Delta_*(\bar{\xi}_1^{p-1}) \tag{5.3}$$

with $\tilde{C} := \mathbb{F}_p[\beta, \beta_2, \ldots] \otimes \Lambda(\alpha_2, \alpha_3, \ldots) \subset H_*(A)$ and $D_*(\beta) = \bar{\xi}_1^p$, $D_*(\beta_i) = \bar{\xi}_i$, $D_*(\alpha_i) = \bar{\tau}_i$ satisfying $\mathscr{P}^1(\alpha_i) = 0$, $Q_0(\alpha_i) = \beta_i$, $\mathscr{P}^1(\beta_i) = \beta_{i-1}^p$, $\mathscr{P}^1\beta_2 = \beta$ and $Q_0(\beta) = \Delta_*(\bar{\xi}_1^{p-1})$, $Q_1(\beta_2) = -\Delta_*(\bar{\xi}_1^{p-1})$.

Define a weight function wt on the monomials in $H_*(A)$ by $\mathrm{wt}(\beta) = \mathrm{wt}(\Delta_*(\bar{\xi}_1^{p-1})) = \mathrm{wt}(\beta_2) = \mathrm{wt}(\alpha_2) = p$, $\mathrm{wt}(\alpha_i) = \mathrm{wt}(\beta_i) = p^{i-1}$ for $i \geq 2$ and $\mathrm{wt}(x \cdot y) = \mathrm{wt}(x) + \mathrm{wt}(y)$. Let $W^A(n)$ be the $\mathbb{F}_p$-vector space generated by all monomials of weight n. Then $Q_0, Q_1, \mathscr{P}^1$ respect weight and $W^A(n)$ is an $\mathscr{A}(1)$-module. The exact sequence (5.2) decomposes into short exact sequences of $\mathscr{A}(1)$-modules (with $\Delta'_*(x) := \Delta_*(x \cdot \bar{\xi}_1^{p-1})$):

$$0 \to \Sigma^{pq-1} W(p(m-1)) \xrightarrow{\Delta'_*} W^A(pm) \to W(pm) \to 0 \tag{5.4}$$

and $H_*(A) \cong \bigoplus_m W^A(pm)$. As a sequence of E-modules (5.4) behaves differently for $m \not\equiv 0(p)$ or $m \equiv 0(p)$. This may be seen either by computing directly the Q_0- and Q_1-homology of $W^A(pm)$ by the long exact sequence induced from (5.4) or using (4.10) of [3] which implies that (5.4) with $n = pm$ is E-isomorphic to

$$0 \to H_*(\Sigma^{qn-1}K(n-1)) \to H_*(\Sigma^{qn-1}X_K(n)) \to H_*(\Sigma^{qn}K(n)) \longrightarrow 0. \tag{5.5}$$

Here $K(n)$ is the integral Brown–Gitler spectrum and $X_K(n)$ is the cofibre of a certain map $F: K(n) \to K(n-1)$. If $m \equiv 0(p)$ then F is of Adams filtration 2, hence (5.5) splits as a sequence of $\mathscr{A}$-modules, so (5.4) must split over E, and $W^A(pm)$ has exactly two non-trivial Q_0- and Q_1-homology classes. The two Q_0-homology classes are represented by $\Delta_*(\bar{\xi}_1^{p-1})\cdot\beta^{m-1}$ and β^m. If $m \not\equiv 0(p)$ then $W^A(pm)$ is a free E-module (see Corollary 5.9 below) and (5.4) does not split.

To extend these results from E to $\mathscr{A}(1)$ we need lifts

$$\tilde{s}_n^{\otimes} : \Sigma^{qn} N_{\otimes}(n) \to H_*(A)$$

of $s_n^{\otimes} : \Sigma^{qn} N_{\otimes}(n) \to H_*(l)$ through $D_* : H_*(A) \to H_*(l)$ if $n \equiv 0(p^2)$. Since $D : A \to l$ is a map of ring spectra it is enough to construct lifts $\tilde{s}_n^{\otimes} : \Sigma^{qn} N_{\otimes}(n) \to H_*(A)$ for $n = sp^a$ and then extend by multiplication.

Definition of $\tilde{s}_n^{\otimes}$ for $n = sp^a$, $a \geq 2$, $0 \leq s < p$: Recall $N_{\otimes}(n) = N(n) \oplus F(n)$ where $F(n)$ is $\mathscr{A}(1)$-free and $N(n)$ is the pure lightning flash module. The definition of $\tilde{s}_n : \Sigma^{qn} N(n) \to H_*(A)$ is parallel to the one of $s_n : \Sigma^{qn} N(n) \to H_*(l)$. Instead of $\bar{\tau}_i$, $\bar{\xi}_i$ we use their preimages α_i, β_i under D_* to define the classes b_i, $\tilde{b}_i$. Then steps (i), (ii), (iv) and (v) remain unchanged, only in step (iii) in fixing a choice for $\tilde{b}_{ps}$ one has to check that b_{ps} is indeed a monomial in $\beta^p, \beta_2, \ldots$ (that there is a choice follows already from our knowledge of the Q_0-homology of $W^A(n)$, as discussed above). That b_{ps} is a monomial in $\beta^p, \beta_2, \ldots$ is clear as long as no β_2 in $\tilde{b}_j$ is involved (since $Q_1(\beta_2) = -\Delta_*(\bar{\xi}_1^{p-1})$, whereas $Q_1(\bar{\xi}_2) = 0$). If all variables β_i, $i > 2$, in b_{jp} are used up, we arrive at $b_{jp} = \beta_2^m$ with $m = sp^{a-1}$. We choose $\tilde{b}_{jp} = \beta_2^{m-1}\cdot\alpha_2$ as preimage under Q_0. Then $Q_1(\beta_2^{m-1}\cdot\alpha_2)$ involves a term with $\Delta_*(\bar{\xi}_1^{p-1})$ but, as is easy to see, the next b_k with $k \equiv 0(p)$ is again a monomial in β^p and β_2. So we can proceed as above.

$\tilde{s}_n$ is defined by $\tilde{s}_n(\tilde{a}_i) = \tilde{b}_i$, $\tilde{s}_n(a_i) = b_i$. Then $D_*\tilde{s}_n = s_n$. Since $F(n)$ is $\mathscr{A}(1)$-free, there is a lift $\tilde{y}_n : F(n) \to W^A(n)$ with $D_* \circ \tilde{y}_n = y_n$. Let $\tilde{s}_n^{\otimes} := \tilde{s}_n + \tilde{y}_n : \Sigma^{qn} N_{\otimes}(n) \to W^A(n)$. For general n with $n = \sum_{i=a}^{b} n_i p^i$, $a \geq 2$, define

$$\tilde{s}_n^{\otimes} := m \circ (\tilde{s}_{n_a p^a}^{\otimes} \otimes \tilde{s}_{n_{a+1} p^{a+1}}^{\otimes} \otimes \cdots \otimes \tilde{s}_{n_b p^b}^{\otimes}),$$

where m is the multiplication in $H_*(A)$. Then clearly $D_* \circ \tilde{s}_n^{\otimes} = s_n^{\otimes}$.

For n with $v(n) \geq 2$ define $S_{\otimes}(n) := \Sigma^{qn-1} N_{\otimes}(n-p) \oplus \Sigma^{qn} N_{\otimes}(n)$ and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{qn-1} N_{\otimes}(n-p) \arrow[r] \arrow[d, "s_{n-q}^{\otimes}"] & S_{\otimes}(n) \arrow[r] \arrow[dd, "g_n^{\otimes}"] & \Sigma^{qn} N_{\otimes}(n) \arrow[r] \arrow[ddl, "\tilde{s}_n^{\otimes}"'] \arrow[dd, "s_n^{\otimes}"] & 0 \\
 & \Sigma^{q-1+(p-1)q} W(n-p) \arrow[d, "\cdot \bar{\xi}_1^{p-1}", "\cong"'] \arrow[dr, "\Delta_*'"'] & & & \\
0 \arrow[r] & \Sigma^{q-1} W(n-1) \arrow[r, "\Delta_*"] & W^A(n) \arrow[r, "D_*"] & W(n) \arrow[r] & 0
\end{tikzcd} \tag{5.6}$$

Define $g_n^{\otimes}$ on $\Sigma^{qn} N_{\otimes}(n)$ by $\tilde{s}_n^{\otimes}$ and on $\Sigma^{qn-1} N_{\otimes}(n-1)$ by $\Delta_*(\bar{\xi}_1^{p-1}\cdot s_{n-p}^{\otimes})$. Then $g_n^{\otimes}$ is an $\mathscr{A}(1)$-surjection with a finite free kernel since $s_n^{\otimes}$ and $s_{n-p}^{\otimes}$ have this property. This determines $W^A(n)$ for $n \equiv 0(p^2)$ as a stable $\mathscr{A}(1)$-module.

Corollary 5.7. *For $n \equiv 0(p^2)$ the short exact sequence of $\mathscr{A}(1)$-modules*

$$0 \longrightarrow \Sigma^{qp-1} W(n-p) \xrightarrow{\Delta'_*} W^A(n) \longrightarrow W(n) \longrightarrow 0$$

splits.

Proof. $\tilde{s}_n^{\otimes} : \Sigma^{qn} N_{\otimes}(n) \to W^A(n)$ may not factorize through $W(n)$, but $\tilde{s}_n^{\otimes}$ restricted to $\ker(s_n^{\otimes})$ has a lift $h_1 : \ker(s) \to \Sigma^{pq-1} W(n-p) \cdot \bar{\xi}_1^{p-1}$. Choose a splitting $\tau : \Sigma^{qn} N_{\otimes}(n) \to \ker(s)$, then $s'_n = \tilde{s}_n^{\otimes} - \Delta_* \circ h_1 \circ \tau$ vanishes on $\ker(s_n^{\otimes})$ and s'_n factorizes through $W(n)$ giving the splitting. $\square$

Consider now n with $v(n) = 1$: The first low-dimensional cases $n = pm$, $1 \le m < p$, are easily checked by a direct calculation, giving $W^A(pm) \cong \Sigma^{qn-1} E_m$. For $n = \sum_{i=1}^{b} n_i p^i$ with $n_1 \ne 0$ let $m = n - n_1 p$ and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{qp-1} W(n_1 p - p) \otimes W(m) \arrow[r, "\Delta'_* \otimes 1"] \arrow[d, "m_1"] & W^A(n_1 p) \otimes W(m) \arrow[r, "D_* \otimes 1"] \arrow[d, "m_2"] & W(n_1 p) \otimes W(m) \arrow[r] \arrow[d, "m_3"] & 0 \\
0 \arrow[r] & \Sigma^{qp-1} W(n-p) \arrow[r, "\Delta'_*"] & W^A(n) \arrow[r, "D_*"] & W(n) \arrow[r] & 0
\end{tikzcd} \tag{5.8}$$

with exact rows. Here m_1, m_3 are defined by multiplication in $H_*(l)$ and m_2 is defined by using multiplication in $H_*(A)$ and an $\mathscr{A}(1)$-section $\tau : W(m) \to W^A(m)$ of D_* (Corollary 5.7). The proof for the commutativity of (5.8) uses the properties of the multiplication of A stated in Lemma 3.6 [3].

From Proposition 4.5 we easily get that m_1 and m_3 are surjective $\mathscr{A}(1)$-maps with finite free kernels. Hence the same is true for m_2. Now $W^A(n_1 p) \cong E_{n_1}$ up to suspension implies by (2.1) that $W^A(n_1 p) \otimes W(m)$ is a direct sum of copies of E_i, $i = 1, \dots, p$. We may split off the kernel of m_2 consisting of a finite sum of E_p's to conclude

Corollary 5.9. *For n with $v(n) = 1$ the $\mathscr{A}(1)$-module $W^A(n)$ is a finite sum of suspensions of $E_1, E_2, \dots, E_p$.*

6. The tagged lightning flash module

For the purpose of the splitting of $A \wedge A$ the information on the $\mathscr{A}(1)$-module structure of $H_*(A)$ in Section 5 is sufficient. However it seems sensible to say something on the internal structure at this place. Since we have no applications of these results, we shall only sketch the arguments. Because of Corollaries 5.7 and 5.9 the structure of $W(n) \subset H_*(l)$ with $n \equiv 0(p^2)$ will complete the picture for $H_*(A)$.

We know that $W(n)$ for $n = sp^a$ consists of a copy of the pure lightning flash module and a finite number of copies of suspensions of $E_p = \mathscr{A}(1)_*$. We are going to show, that for general n, $W(n)$ contains a module $G'(n)$ which we call tagged lightning flash module. This is the only new indecomposable $\mathscr{A}(1)$-module which appears, the other indecomposable modules in $W(n)$ are E_i's with $1 \le i \le p$. Note that there are many other indecomposable $\mathscr{A}(1)$-modules which are not of this type.

We begin by defining the stable class $G(n)$ of the tagged lightning flash module $G'(n)$. This will determine $G'(n)$ up to isomorphism (2.5). Write

$$n = \sum_{i=1} n_i p^i = s + t(p-1), \quad 0 \le s < p-1, \ 0 \le n_i < p$$

as

$$n = J_0 + J_1 + \cdots + J_t,$$

where J_i is for $i > 0$ a sum of exactly $p-1$ powers of p,

$$J_i = p^{j_{i,1}} + p^{j_{i,2}} + \cdots + p^{j_{i,p-1}}$$

and

$$J_0 = p^{j_{0,1}} + p^{j_{0,2}} + \cdots + p^{j_{0,s}}.$$

To be specific we shall work from higher to smaller p powers, so that the sequence of exponents $\ldots, j_{i,1}, \ldots, j_{i,p-1}, j_{i+1,1}, \ldots$ will be increasing (but see also Remark 1 at the end of Section 6). Define $G(n)$ to be

$$G(n) := N(J_0) \otimes N(J_1) \otimes N(J_2) \otimes \cdots \otimes N(J_t). \tag{6.1}$$

If we restrict from $\mathscr{A}(1)$ to Γ-modules, then for $k > 0$, $N(J_k) \cong V_{p-1}$ (up to free modules and suspensions) and $N(J_0) \cong V_s + V_{s+1}$ (up to free modules and suspensions). From Theorem 3.1 we see that

$$G(n) \cong V_s \oplus V_{s+1} \quad \text{or} \quad V_{p-s} \oplus V_{p-s-1}$$

again up to free modules and suspensions.

This already shows that $G(n)$ contains no copies of E_i, $i < p$, as direct summands and determines the stable class of $E_j \otimes^{\Delta} G(n)$ over $\mathscr{A}(1)$ by (2.1).

That $N(J_k)$ is stably isomorphic to $\Sigma^l V_{p-1}$ as Γ-module means that $N(J_k)$, $k > 0$, has only one non-trivial $\mathscr{P}^i$-homology class. Then the same proof as given for E in [1] applies to prove

Lemma 6.2. *$N(J_k)$, $k > 0$, is stably invertible.*

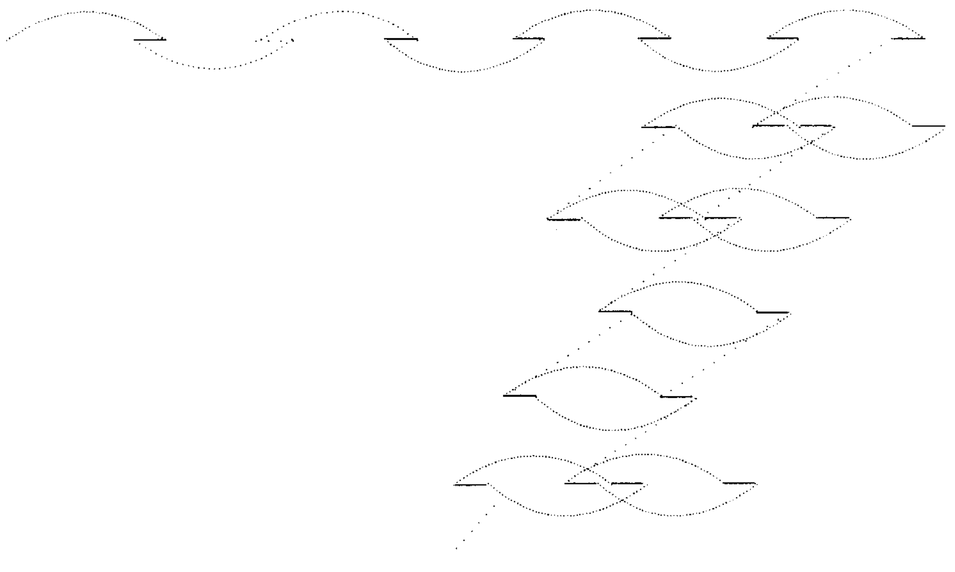
Corollary 6.3. *$G(n)$ is stably indecomposable.*

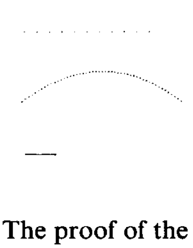
Proof. Any stable decomposition of $G(n)$ would yield a stable decomposition of $N(J_0)$ by tensoring $G(n)$ with the inverse classes of the $N(J_i)$, $i = 1, \ldots, t$. But $N(J_0)$ is a pure lightning flash module which is indecomposable already over E. $\square$

Hence we may write $G(n) = G'(n) \oplus F$ where F is free and $G'(n)$ contains no free summands. Observe that each E_i, $1 \le i \le p$, is indecomposable since it is a cyclic $\mathscr{A}(1)$-module.

Remark 6.4. The internal structure of the tagged lightning flash module $G'(n)$ is harder to obtain. Since we do not need it, we shall only refer to the result: $G'(n)$ consists of a pure lightning flash module $N(n)$ together with a string of E_i's – the tag – which are all linked by a non-trivial $\mathscr{P}^1$-action. The shape of the tag depends only on the length $l(n) := \sum_{i=1} n_i$ and the type $t := m_0$ ($m = v(n!) = \sum m_i p^i$). The number of E_i's contained in the tag is given by $(l - m_0)/p$ and the succession of E_i's in the tag is periodic with period $2(p-1)$ and the way in which the E_i's in the tag start depends only on m_0. The tag is tied to $N(n)$ by a $\mathscr{P}^1$ at the place where the $\mathscr{P}^1$-homology class of $N(n)$ sits, that is at the upper end. See picture 6.5.

6.5. An example of the tagged lightning flash module for $p = 3$:





The proof of the next proposition – which is a long but straightforward exercise – is left to the reader.

Proposition 6.6. *For $s < p$ there is a stable $\mathscr{A}(1)$-isomorphism*

$$\Sigma^{qm}(N(p^{a_1} + p^{a_2} + \cdots + p^{a_s})) \oplus E(a_1, a_2, \ldots, a_s)$$
$$\cong W(p^{a_1}) \otimes W(p^{a_2}) \otimes \cdots \otimes W(p^{a_s})$$

where $E(a_1, a_2, \ldots, a_s)$ is a finite direct sum of suspensions of E_i's and $m = p^{a_1} + p^{a_2} + \cdots + p^{a_s}$.

Corollary 6.7. *For $n < p$ there is a stable $\mathscr{A}(1)$-isomorphism*

$$E(n) \oplus W(np^a) \cong W(p^a) \otimes \cdots \otimes W(p^a) \quad (n \text{ copies})$$

where $E(n)$ is a finite direct sum of suspensions of E_i's.

Proof. This follows from Proposition 6.6 with $\Sigma^{qnp^a} N(np^a) \cong W(np^a)$ (modulo free $\mathscr{A}(1)$-modules). $\square$

We now have control on what is happening if we change the tensor product presentation of $W(n)$. Stably:

$$W(n) \cong W(n_a p^a) \otimes \cdots \otimes W(n_b p^b), \quad n = \sum_{i=a}^{b} n_i p^i, \quad a \geq 1.$$

We decompose the $W(n_i p^i)$-factors using Corollary 6.7 and then recollect the $W(p^j)$-factors into $N(J_0) \otimes N(J_1) \otimes \cdots \otimes N(J_s)$ using Proposition 6.6. This gives stable isomorphisms

$$W(n) \oplus E_1(n) \cong W(p^a) \otimes \cdots \otimes W(p^b) \cong \Sigma^{nq} N(J_0) \otimes \cdots \otimes N(J_s) \oplus E_2(n),$$

where $E_1(n)$ and $E_2(n)$ are finite sums of suspensions of E_i's. Now $N(J_0) \otimes N(J_1) \otimes \cdots \otimes N(J_t) = G(n)$ is the stable tagged lightning flash module and by the uniqueness of the decomposition into indecomposable modules we have determined the stable (and unstable) decomposition of $W(n)$ into indecomposables.

Proposition 6.8. *For $n \equiv 0(p)$ the $\mathscr{A}(1)$-module $W(n) \subset H_*(l)$ is isomorphic to $G'(n) \oplus E'(n)$ where $E'(n)$ is a finite direct sum of suspensions of E_i's, $1 \leq i \leq p$.*

Remarks. (1) If we recollect p-powers in n in a way different to the one above, we get an indecomposable stable module $\bar{G}(n)$. The argument above gives stable equivalences

$$\bar{G}(n) \oplus E_3(n) \cong W(n) \oplus E_1(n) \cong G(n) \oplus E_2(n)$$

with $E_i(n)$ as above. But $G(n)$ and $\bar{G}(n)$ have at most two non-trivial $\mathscr{P}^1$-homology classes, whereas E_j, $j \neq p$, has at least three non-trivial $\mathscr{P}^1$-homology classes. Hence by the uniqueness in the decomposition theorem we get $\bar{G}(n) \cong G(n)$ and the definition of the tagged lightning flash module does – up to isomorphism – not depend on the way of recollecting p-powers in n.

(2) A geometric realization of the modules E_i, $N_{\otimes}(n)$ is known, also a stable realization of $G'(n)$, i.e. a realization of $G'(n) \oplus F(n)$ where $F(n)$ is free. I do not know a geometric model for $G'(n)$, such models would yield an A-module splitting of $A \wedge A$ into indecomposable pieces.

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